B3 - Probability III : Mid-Semester Exam

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09th September 2025. **Time Limit : 2.5 hours.** **Max. points : 30.**

There are two parts to the question paper - PART A and PART B. Read the instructions for each section carefully.

1 PART A : MULTIPLE-CHOICE QUESTIONS - 10 Points.

Please write only the correct choice(s) (for ex., (a), (b) et al.) in your answer scripts. No explanations are needed. Write PART A answers in a separate page.

Some questions will have multiple correct choices. Answer all questions. Each question carries 2 points. 1 point will be awarded if only some correct choices are chosen and no wrong choices are chosen.

1. Let $\Omega = \{1, 2, 3, 4\}$ and consider $\mathcal{F} := \sigma(\{1, 2\}, \{2, 3\}, \{3, 4\})$.
   (a) $\mathcal{F}$ has at most 14 elements.
   (b) $\mathcal{F}$ contains at least 12 elements.
   (c) $\mathcal{F}$ is the power-set.
   (d) $\{4\} \notin \mathcal{F}$.

2. Which of the following are Borel-measurable functions from $\mathbb{R}$ to $\mathbb{R}$?
   (a) $\mathbf{1}_{\mathbb{Q}}$ where $\mathbb{Q}$ is the set of rationals.
   (b) $\mathbf{1}_A$ where A^c is countable.
   (c) $\mathbf{1}_A$ for $A \subset \mathbb{R}$.
   (d) $\mathbf{1}_A \mathbf{1}_{\mathbb{Q}}$ where $\mathbb{Q}$ is the set of rationals and $A \subset \mathbb{R}$.

3. Let $\mathbb{P}$ be a probability measure on $(\mathbb{R}, \mathcal{B})$. Which of the following must be true ?
   (a) $\mathbb{P}(\{n, n+1, \ldots, \}) = 1$ for all $n \in \mathbb{N}$.
   (b) $\mathbb{P}(\{q\}) = 0$ for all $q \in \mathbb{Q}$.
   (c) $\mathbb{P}(\{-n, -n-1, \ldots, \}) = 1$ for all $n \in \mathbb{N}$.
   (d) For all $\epsilon > 0$, there exists a compact subset $K \subset \mathbb{R}$ such that $\mathbb{P}(K) \geq 1 - \epsilon$.

4. For which of the following conditions, is there a probability measure $\mathbb{P}$ on $([0, 1], 2^{[0,1]})$ satisfying the condition.

(a) $\mathbb{P}[0,1/2] = \mathbb{P}[1/2,1] = 2/3$.

(b) $\mathbb{P}[0,1/2) = \mathbb{P}(1/2,1] = 2/3$.

(c) $\mathbb{P}[0,1/2) = \mathbb{P}(1/2,1] = 0$.

(d) $\mathbb{P}[0,1/2] = \mathbb{P}[1/2,1] = 1/3$.

5. Which of the following functions can be a CDF ?

   (a) $F(x) = \min\{x_+, 1\}, x \in \mathbb{R}$.

   (b) $F(x) = \min\{\sqrt{x_+}, 1\}, x \in \mathbb{R}$.

   (c) $F(x) = \frac{1}{2} + \frac{1}{\pi}\arctan(x), x \in \mathbb{R}$

   (d) $F(x) = e^{-e^{-x}}, x \in \mathbb{R}$.

2 PART B : 20 Points.

Answer any two questions only. All questions carry 10 points.

Give necessary justifications and explanations for all your arguments. If you are citing results from the class, mention it clearly.

1. Let X be a Poisson random variable with parameter λ. Find k so that $\mathbb{P}(X = k)$ is maximal and also show that $\mathbb{E}(Xf(X)) = \lambda\mathbb{E}(f(X+1))$ for any bounded function $f : \mathbb{N} \cup \{0\} \to \mathbb{R}$.

2. Let X be a random variable with pdf $f(x) = cx(1-x)$ if $0 \le x \le 1$ and $f(x) = 0$ otherwise. Find a measurable function $G : [0,1] \to [0,1]$ such that $G(U)$ has the same CDF as the random variable $X(1-X)$.

3. Let $b \in (0,\infty)$ and X be a random variable with pdf $f(x) = \frac{1}{2b}e^{-\frac{|x|}{b}}$, $x \in \mathbb{R}$.

   (a) Compute the moment generating function and all the moments of X. (**6**)

   (b) Find the pdf and CDF of the random variable $|X|$. (**4**)

Notations

1. $\mathbb{N}$ - set of Natural numbers; $\mathbb{Q}$ - set of Rationals ; $\mathcal{B}$ - Borel σ-algebra.

2. pmf - probability mass function; pdf - probability density function; CDF - Cumulative distribution function.

3. U is a Uniform$[0,1]$ random variable ; X is a Poisson random variable with parameter λ if the pmf is $\lambda^k e^{-\lambda}/k!, k = 0,1,2\ldots$.

4. For a random variable X, the kth moment is $\mathbb{E}\left(X^k\right)$ and the moment generating function is $\mathbb{E}\left(e^{tX}\right), t \in \mathbb{R}$.